



Algebraic Multigrid for Maxwell's Equations

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Maxwell's Equations

Eddy Current Formulation:

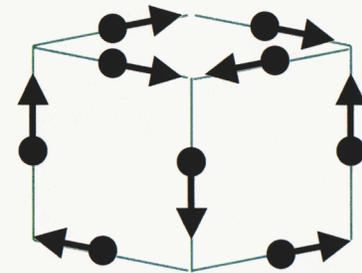
$$\begin{aligned} \nabla \times \mathbf{H} &= \mathbf{J} \text{ in } \Omega & \nabla \cdot \mathbf{B} &= 0 \text{ in } \Omega & \nabla \cdot \mathbf{J} &= 0 \text{ in } \Omega \\ \mathbf{B} &= \mu \mathbf{H} & \nabla \times \mathbf{E} &= \frac{\partial \mathbf{B}}{\partial t} \text{ in } \Omega & \mathbf{J} &= \sigma \mathbf{E} \end{aligned}$$

Discretizing in time, substituting

$$\int \sigma \mathbf{E}^{n+1} \cdot \hat{\mathbf{E}} + \int \frac{1}{\mu} \Delta t \text{curl} \mathbf{E}^{n+1} \cdot \text{curl} \hat{\mathbf{E}} = \dots$$

Using edge elements:

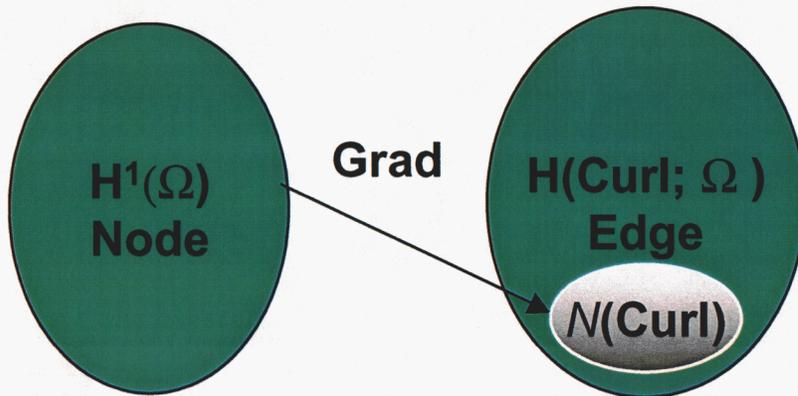
$$M_h + S_h = \dots$$



⇒ Large null space of (curl, curl) can dominate mass !!

Edge elements & the null space

Portion of De Rham complex:



$$\text{Null}(\text{curl}, \text{curl}) = \{\nabla\phi \mid \phi \text{ scalar}\}$$

Discrete null space, T_h , is an $N_e \times N_v$ matrix:

$$\forall v \in V, \forall e \in E_v \quad T_h(e, v) = \begin{cases} +1 & \text{if } v = \text{head}(e) \\ -1 & \text{if } v = \text{tail}(e) \\ 0 & \text{otherwise} \end{cases}$$

- Discrete grad T_h : nodes \rightarrow edges.
- $S_h T_h = \emptyset$.
- Large null space.

Multigrid

MG(b, u, k):

$$R_k(A_k, b_k, u_k)$$

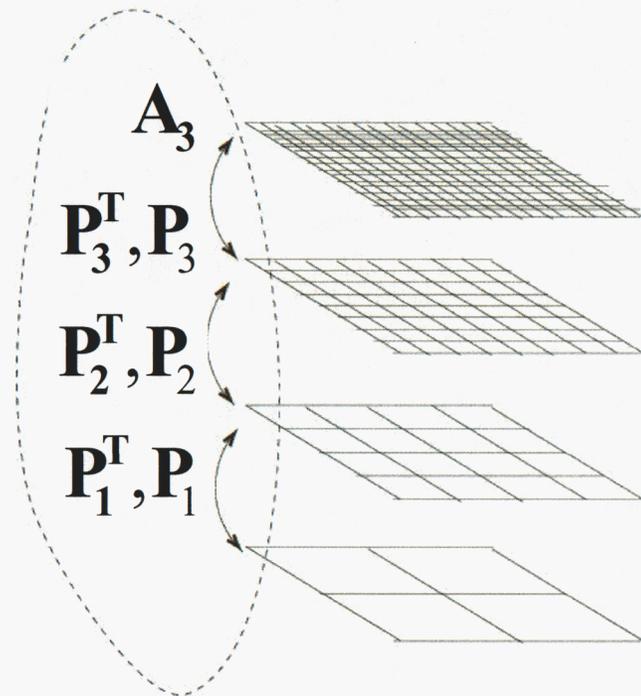
if ($k \neq 0$)

$$r_k = b_k - A_k u_k$$

$$b_{k-1} = (P_k)^T r_k; u_{k-1} = 0$$

$$MG(b_{k-1}, u_{k-1}, k-1)$$

$$u_k = u_k + P_k u_{k-1}$$



A_f : fine grid discretization, $S_h + iM_h$

A_{k-1} : $(P_k)^T A_k P_k$

R_k : relaxation/smoothers

P_k : interpolation

$(P_k)^T$: restriction

Distributed Relaxation

(Brandt, Vassilevski/Wang, Hiptmair)

Relax[†] (e.g. sym-GS): $Ax_e = f$.

⇒ Damps high frequency error in Range of S_h !!!

Distributed Relaxation

(Brandt, Vassilevski/Wang, Hiptmair)

Relax[†] (e.g. sym-GS): $Ax_e = f$.

Relax[†] : $A_{\text{proj}}x_n = T^*(f - Ax_e)$.

Correct edge solution: $x_e = x_e + Tx_n$.

Relax: $Ax_e = f$

where $A_{\text{proj}} = T^T A T$

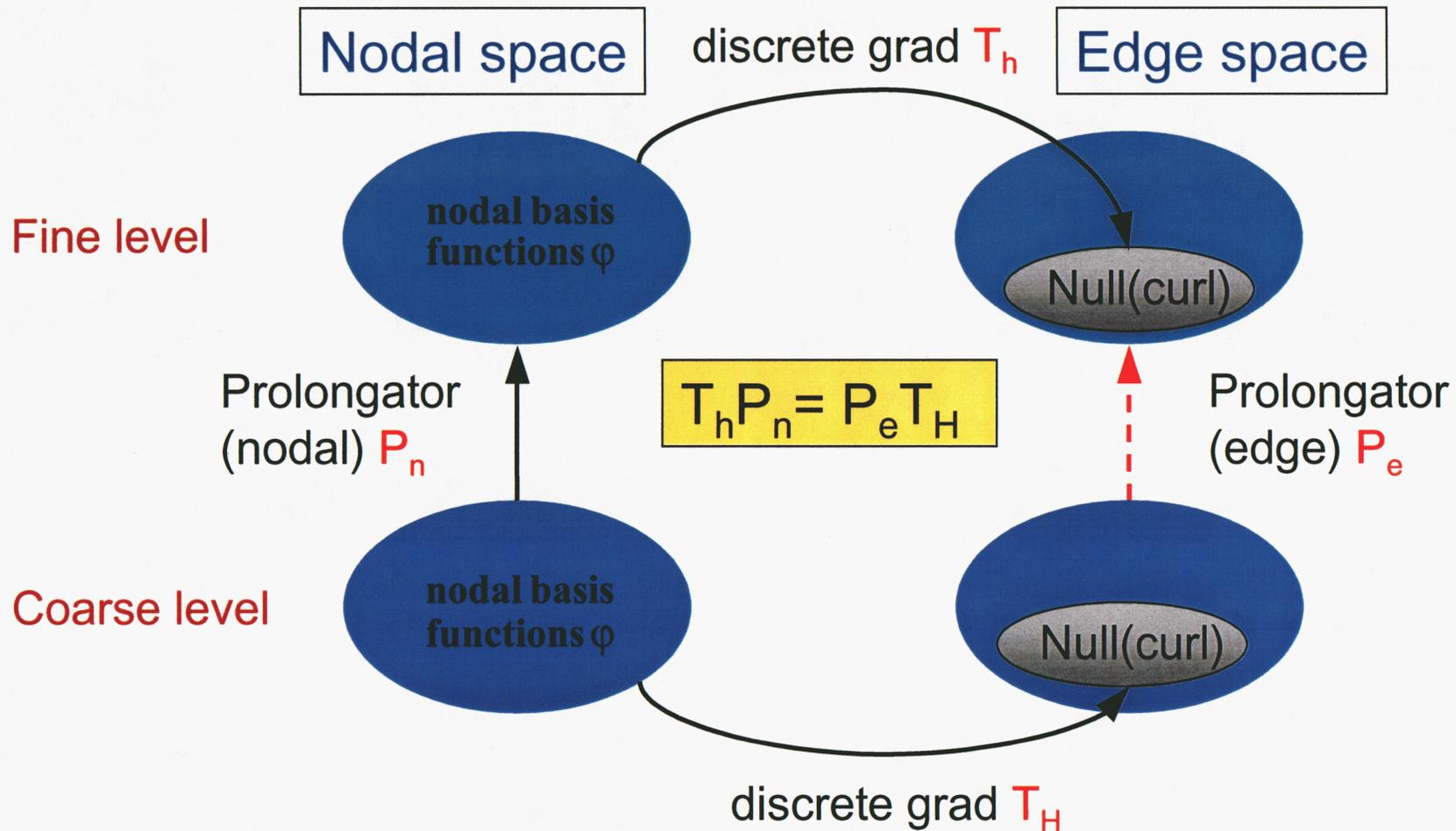
Central idea: Smooth each part of $\text{Null}(\text{curl}) \oplus \text{Null}(\text{curl})^\perp$.

[†]we use Chebyshev polynomials.

Alternative: (Arnold/Falk/Winther) Overlapping Schwarz



Coarse Grid Correction (R/S)



Note: $S_H = (P_e)^T S_h P_e$ & $T_h P_n = P_e T_H$
 $\Rightarrow S_H T_H = \emptyset$

Smoothing Prolongator

Energy reduction:

$${}^sP_e = (I - \rho D^{-1}S_h)P_e, \quad D = \text{diag}(S_h)$$

preserves commuting:

$$\begin{aligned} {}^sP_e T_H &= (I - \rho D^{-1} S_h) P_e T_H \\ &= (I - \rho D^{-1} S_h) T_h P_n \\ &= T_h P_n \end{aligned}$$

Grid Size	P_e	sP_e	$\text{sIs}P_e$
27×27	15/.04	11 / .02	9 / .02
243×243	96 / 28.1	25 / 5.4	18 / 4.6
729×729	297 / 800.7	39 / 76.0	23 / 52.7

P_e = R/S interpolant

sP_e = smoothed R/S interpolant

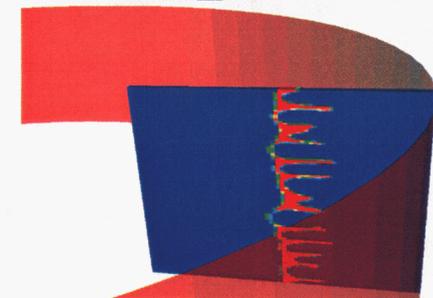
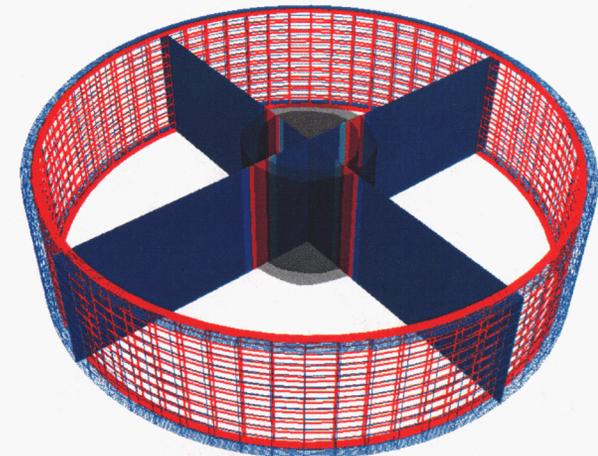
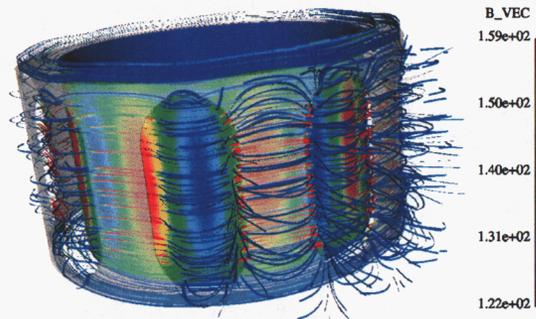
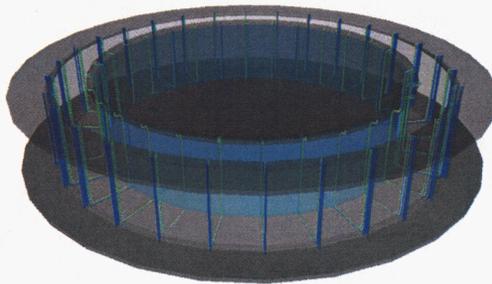
$\text{sIs}P_e$ = smoothed least sq. interpolant

$\sigma = 10^{-3} \quad V(1,1)$

ALEGRA-HEDP Simulation

procs	# Elements	Iter	CPU (s)
4	37,120	37	7
16	155,904	43	10
50	492,912	49	16
128	1,259,976	65	27
240	2,457,650	77	53

σ varies from 10^{-3} to 10^3 , ASCI White
CG with simple preconditioners fails.



Frequency Domain

$$\nabla \times \nabla \times \mathbf{E} + i\omega\mu\sigma \mathbf{E} = \mathbf{rhs}$$

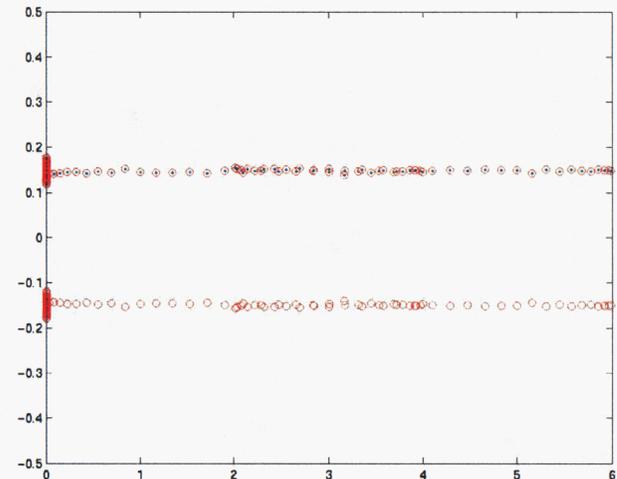
Instead of $S + iM$ consider equivalent real form (ERF):

$$\tilde{A} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} S & -M \\ M & S \end{pmatrix}$$

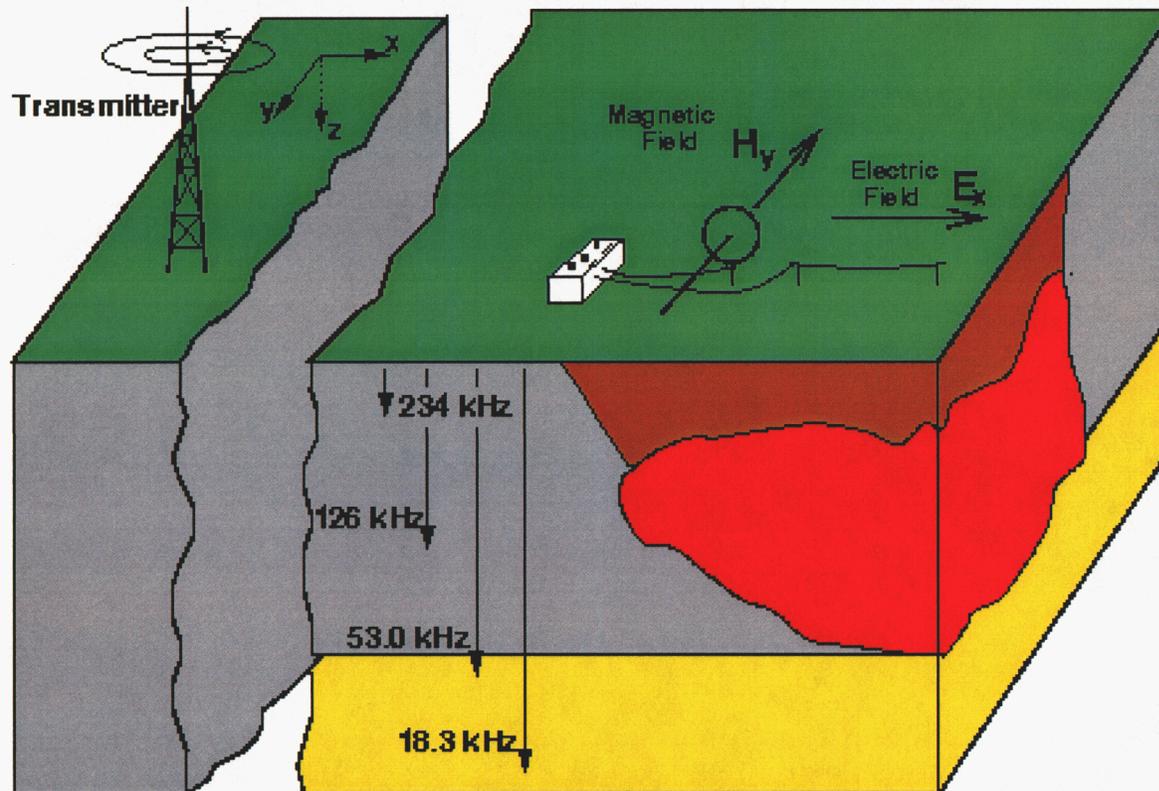
where

$$\begin{aligned} u_1 &= \text{Re}(E) & b_1 &= \text{Re}(\mathbf{rhs}) \\ u_2 &= \text{Im}(E) & b_2 &= \text{Im}(\mathbf{rhs}) \end{aligned}$$

*Note: ERF reflects eigenvalues about real axis
⇒ Care must be taken!!!*



The Radio Magnetotelluric Method



Impedance measurement $Z_{xy} = E_x / H_y$ used to infer subsurface electrical properties

Multigrid & Equivalent Real Forms (ERFs)

General idea:

Mimic $S + iM$ algorithm with equivalent real form!

1. Apply AMG scheme to $S \Rightarrow P_k, S_k, T_k$
2. Define ERF operators:

$$\tilde{P}_k = \begin{pmatrix} P_k & \\ & P_k \end{pmatrix}, \quad \tilde{T}_k = \begin{pmatrix} T_k & \\ & T_k \end{pmatrix}$$

$$\tilde{T}_k^T \tilde{A}_k \tilde{T}_k = \begin{pmatrix} T_k^T S_k T_k & -T_k^T M_k T_k \\ T_k^T M_k T_k & T_k^T S_k T_k \end{pmatrix} = \begin{pmatrix} & -T_k^T M_k T_k \\ T_k^T M_k T_k & \end{pmatrix}$$

$$\tilde{A}_{k-1} = \tilde{P}_k^T \tilde{A}_k \tilde{P}_k = \begin{pmatrix} S_{k-1} & -M_{k-1} \\ M_{k-1} & S_{k-1} \end{pmatrix}, \quad M_{k-1} = P_k^T M_k P_k$$

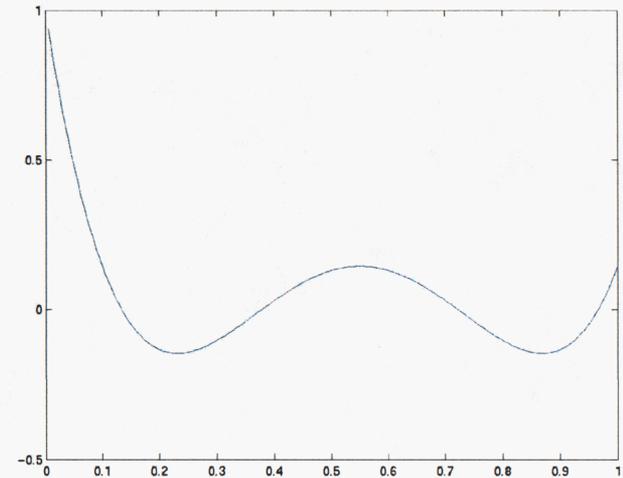
ERF Distributed Relaxation

- $\tilde{T}_k^T \tilde{A}_k \tilde{T}_k$ relaxation decouples into 2 independent problems

– Apply Chebyshev to each

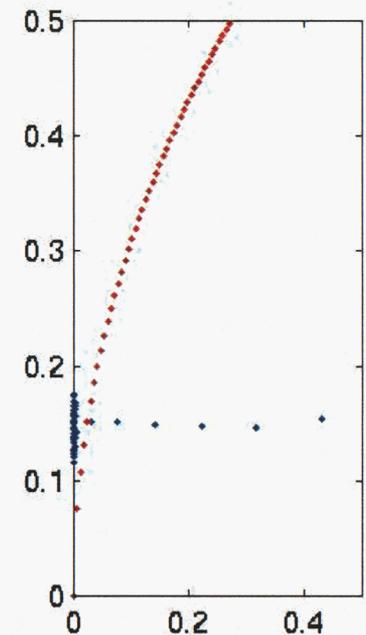
$$x^{(i)} = x^{(i-1)} + p(\tilde{A}_{proj})(b - \tilde{A}_{proj}x^{(i-1)})$$

where $p()$ damps high frequencies.



- Apply complex Chebyshev to $S_k + i M_k$ without complex arith.

Apply Chebyshev ~~to~~ ERF or real Chebyshev ~~to~~ $S_k + i M_k$



Preliminary Results

1 problem on a cube: 24,000 edges & 8000 nodes

M	Iterations
none	-
Distributed relaxation	78
MG	23

8 processors using GMRES, $\|r\|$ reduced by 10^{10} .

Conclusions

- Null space is key for Maxwell's Equations.
- Discretization: edge elements.
- Coarse grid correction:
 - Build P_n via traditional AMG.
 - Derive P_e to maintain commutative property.
 - Improve P_e to reduce energy.
- Use Equivalent Real Forms for Frequency Domain
 - Mimic complex arithmetic algorithm with ERF algorithm
 - Distributed smoother
 - Complex polynomials on ERF system
 - 2 decoupled real projects for projected ERF system
- Applications: Z-pinch & subsurface imaging.